THE ASYMPTOTIC BEHAVIOR OF A VORTEX FAR AWAY FROM A BODY IN A PLANE FLOW OF VISCOUS FLUID

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The flow of a plane stream of incompressible viscous fluid past a body is considered, and the velocity field far away from the body is analyzed. Asymptotic formulas are derived for the vortex and the velocity field.

The problem of the asymptotic behavior of a viscous fluid has been known for a long time and had attracted the attention of many researchers. Numerous works of Finn and his disciples are known. Filon [1, 2] had examined the plane case by using the Oseen approximation. He obtained a divergent integral for the moment acting on the body, which is known as Filon's "paradox". Investigations by Goldstein [3, 4], Imai [5], Smith [6], and Finn and Smith [7] followed. The subject of the present paper arose in the course of development of an algorithm for the numerical solution of the problem of flow of a viscous fluid past a circular cylinder.

1. Let S be a cross section of the body and C – a smooth Jordan curve – the boundary of S; the complement of S to the whole plane will be denoted by G. Let (x, y) be rectangular coordinates with origin within S. Let $1 + u$, v be dimensionless velocity components, p the dimensionless pressure, ρ the density, $\rho = 1$. We denote by w the complex velocity $w = v + iu$, by ω the vortex, $\omega = \partial v / \partial x - \partial u/\partial y$, and by R the Reynolds number with $\lambda = R/2$. We shall consider those solutions of the flow problem which satisfy conditions

$$w \in C^3(G) \cap C^1(\overline{G}), \quad \int_G \omega^2\, dx\, dy < \infty \tag{1.1}$$

$$|w| = O(r^{-1/4-\varepsilon}) \qquad (r = \sqrt{x^2 + y^2}) \tag{1.2}$$

where $\varepsilon > 0$ is an arbitrarily small quantity. We set

$$z = x + iy, \qquad \zeta = \xi + i\eta, \qquad \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Let

$$l_0(z) = e^{\lambda x} K_0(\lambda r), \quad m_0(z) = (\bar z / r) e^{\lambda x} K_1(\lambda r), \quad l_0^*(z) = m_0(z) - 1/\lambda z$$

where K_j $(j = 0,1)$ is the MacDonald function. Using these conditions it can be easily shown that

$$\omega(\zeta) = j_0 + \frac{\lambda}{\pi}\int_G \omega\left[u\frac{\partial l_0(\zeta - z)}{\partial x} + v\frac{\partial l_0(\zeta - z)}{\partial y}\right] dx\, dy \tag{1.3}$$

$$j_0 = \frac{1}{2\pi}\int_C \left[\omega\frac{\partial l_0(\zeta - z)}{\partial n} + 2\lambda p\frac{\partial l_0(\zeta - z)}{\partial s}\right] ds \tag{1.4}$$

where n is the outward normal to $\partial S = C$ and s is the length of arc along C.

For the convolution of the two functions f and l we introduce the following notation

$$(f*l)(\zeta) = f*l = \frac{1}{\pi}\int_G f(z)\, l(\zeta - z)\, dx\, dy$$

With this notation

$$w(\zeta) = f(\zeta) + {}^1/_2\lambda i\,(w^2 * k + w^2 * k^*) \qquad (1.5)$$

where

$$k(z) = \frac{\partial}{\partial \bar z} l_0(z), \quad k^*(z) = \frac{\partial}{\partial z} l_0^*(z), \quad f(\zeta) = -\frac{1}{4\pi}\int_C \{l_0(\zeta - z)\times$$
$$\times [(2\lambda p + i\omega)\,dz + dz] - l_0^*(\zeta - z)[(2\lambda p - i\omega)\,dz + d\bar z]\} + g(\zeta) \qquad (1.6)$$

and $g(\zeta)$ is regular in region G.

2. Let

$$L(z) = \begin{cases} r^{-\alpha}\exp[\mu(x - r)] + \delta r^{-2}, & r \geqslant 1 \\ r^{-\alpha_0}, \quad 0 \leqslant \alpha_0 < 2, & r < 1 \end{cases} \qquad (2.1)$$

where $\delta = 1$ or 0. We set

$$\varphi(z) = (|\log r| + 1)^{\beta_0}(r + 1)^{-\beta}, \beta_0 \geqslant 0, \qquad \beta > 0, \qquad J(\zeta) = (\varphi * L)(\zeta)$$

The following Lemmas provide an estimate of the convolution $J(\zeta)$, and are adduced without proof owing to space limitation.

Lemma 2.1. If $\beta \leqslant 1$, $\alpha + \beta > {}^3/_2$, then

$$J(\zeta) < C\varphi(\zeta)\,[\rho^{3/2-\alpha}\Delta_{1,\beta}(\zeta) + \delta\log\rho + \Delta_{3/2,\alpha}(\zeta)] \qquad (2.2)$$

$\rho = |\zeta| \quad \Delta_{\alpha,\beta}(\zeta) = (\log\rho)^{\delta_{\alpha\beta}}$, $\delta_{\alpha\beta}$ — is the Kroneker delta.

We set

$$\sigma(z) = r - x + 1$$

Lemma 2.2. If $1 < \beta \leqslant 2$, $\alpha + \beta > {}^3/_2$, then

$$J(\zeta) < C\varphi(\zeta)\,[\rho^{3/2-\alpha} + \Delta_{3/2,\alpha}(\rho) + \rho^{1-\alpha-\beta/2}\sigma^{1/2(1-\beta)}(\zeta)\Delta_{2\beta}(\rho) + \delta\log\rho] \qquad (2.3)$$

Constant C depends on μ and α, β.

Let

$$\psi(z) = (\log r + 1)^{\beta_0}\begin{cases} r^{-\beta}e^{\mu(x-r)}, & r > 1 \\ r^{-\beta_1}, \quad 0 < \beta_1 < 2, & r < 1 \end{cases} \qquad (2.4)$$

Let us consider convolution

$$J_1(\zeta) = (\psi * L)(\zeta)$$

Lemma 2.3. Inequality

$$J_1(\zeta) < C\{\exp[\mu(\xi - \rho)]\,[\rho^{3/2-\alpha-\beta} + \rho^{-\alpha}\Delta_{3/2,\beta}(\zeta) + \rho^{-\beta}\Delta_{3/2,\alpha}(\zeta)] +$$
$$+ \delta\rho^{-\beta}\sigma^{-1/2}(\zeta)\log\rho + \delta\rho^{-2}\Delta_{3/2,\beta}(\rho)\}(\log\rho)^{\beta_0} \qquad (2.5)$$

is valid.

3. Let $f(z)$ be continuous for $|z| \geqslant R$ and $\lim f(z) = 0$ when $|z| \to \infty$. For $|z| \geqslant R$ function $\Phi(r) = \max|f(z)|$ is determined at $|z| \geqslant r$. We shall call the expression

$$\varliminf_{r\to\infty}\left(\frac{1}{\log r}\log\frac{1}{\Phi(r)}\right)$$

the power order of decrease of function $f(z)$ and denote it by $\delta = \delta(f)$. Assumption (1.2) is written in the form $\delta(w) > {}^1/_4$. We set

$$k(z) - k^*(z) = -L_{11}(z) - iL_{12}(z)$$
$$k(z) + k^*(z) = -iL_{12}(z) + L_{22}(z)$$

Using the asymptotic formulas for Bessel functions, we obtain the estimate

$$|L_{lm}(z)| \leqslant C\lambda \begin{cases} (\lambda r)^{-\alpha_{lm}} e^{\mu(x-r)} + (\lambda r)^{-2}, & \text{if } \lambda r \geqslant 1 \\ (\lambda r)^{-1}, & \text{if } \lambda r < 1 \end{cases} \tag{3.1}$$

where $\mu / \lambda \geqslant \vartheta_0 > 0$, and ϑ_0 is an absolute constant, while C is a constant dependent on ϑ_0 only. Values α_{lm} are

$$\alpha_{11} = \alpha_{22} = {}^3/_2, \qquad \alpha_{21} = 1, \qquad \alpha_{12} = 2 \tag{3.2}$$

Proposition 3.1. If $\delta(w) \leqslant {}^1/_2$, then $\delta(v) \geqslant 2\delta(w)$.

Proof. From (1.5) follows that

$$v(\zeta) = \mathrm{Re} f(\zeta) + \lambda(vu) * L_{11} + {}^1/_2\lambda(v^2 - u^2) * L_{12} \tag{3.3}$$

Taking into consideration estimates (3.1) amd relationships (3.2), by Lemma 2.1 we have

$$v(\zeta) = \mathrm{Re} f(\zeta) + O(\rho^{-2\delta(w)+\varepsilon}) \tag{3.4}$$

where $\varepsilon > 0$ is arbitrarily small. Since $\mathrm{Re} f(\zeta) = O(\rho^{-1})$, we have from the last inequality $\delta(v) \geqslant 2\delta(w)$, Q.E.D.

Proposition 3.2. The estimate $\delta(u) \geqslant {}^1/_2$ is valid.

Proof. By formula (1.5)

$$u(\zeta) = \mathrm{Im} f(\zeta) + \lambda(vu) * L_{21} + {}^1/_2\lambda(v^2 - u^2) * L_{22} \tag{3.5}$$

When $\delta(w) + \delta(v) = \delta(u) + \delta(v) \leqslant 1$, then by Lemma 2.1

$$u(\zeta) = \mathrm{Im} f(\zeta) + O(\rho^{1/2 - \delta(u) - \delta(v) + \varepsilon} + \rho^{-2\delta(u)+\varepsilon})$$

where $\varepsilon > 0$ is arbitrarily small. Since $\delta(v) \geqslant 2\delta(w) > {}^1/_2$, it follows from this that $\delta(u) = \delta(\mathrm{Im} f)$. Hence $\delta(u) \geqslant {}^1/_2$, which contradicts our assumption that $\delta(w) + \delta(v) \leqslant 1$. Thus, $\delta(u) + \delta(v) > 1$, and by Lemma 2.2

$$u(\zeta) = \mathrm{Im} f(\zeta) + O[\rho^{-1/2[\delta(u)+\delta(v)]+\varepsilon} + \rho^{-2\delta(u)+\varepsilon} + \rho^{-1/2-(u)+\varepsilon}] \tag{3.6}$$

From this follows inequality $\delta(u) \geqslant {}^1/_2$, Q.E.D.

Using the asymptotic formulas for Bessel functions, we obtain

$$f(\zeta) = i a_{1/2} \rho^{-1/2} e^{\lambda(\xi - \rho)} + O(\rho^{-1}), \quad a_{1/2} = \left(\frac{\lambda}{2\pi}\right)^{1/2} \int\limits_C \left(p dy - \frac{\omega}{2\lambda} dx\right) \tag{3.7}$$

The derived integral differs from (the expression for) drag by a factor only, hence it is not zero. A rigorous proof of this was given by Smith in [6].

Thus $\delta(u) = \delta(w) = {}^1/_2$, and setting

$$w_{1/2}(\zeta) = i a_{1/2} \rho^{-1/2} e^{\lambda(\xi - \rho)}$$

by virtue of (3.4) and (3.6) we obtain

$$v(\zeta) = O(\rho^{-1+\varepsilon}), \qquad u(\zeta) = \mathrm{Im}\, w_{1/2}(\zeta) + O(\rho^{-3/4+\varepsilon}) \tag{3.8}$$

Relationship (3.8) will be further refined by a rational application of the iteration process to the nonlinear equation (1.5). As the result we obtain a few of the first terms of asymptotics of w, differing by their order of decrease. The orders of decrease form a series of numbers ${}^1/_2$, 1, ${}^3/_2$, 2, ... In progressing through this series, terms containing logarithmic factors will appear in abundance in the asymptotics.

The sum of terms whose order of decrease does not exceed α will be denoted by w_α and it will be assumed that

$$w = w_\alpha + w^{(\alpha + 1/2)}$$

Proposition 3.3. Estimate $\delta(w^{(1)}) \geqslant 1$ is valid.

Proof. We set $w^{(1)} = v^{(1)} + iu^{(1)} = v + iu^{(1)}, \quad f = f_{1/2} + f^{(1)}, \quad f_{1/2} = w_{1/2}$

Then relationship (1.5) yields

$$w^{(1)} = f^{(1)} + {}^1/_2\,\lambda i\,[w_{1/2}^2 * (k + k^*) + 2\,(w_{1/2} w^{(1)}) * k - 2\,(w_{1/2}\overline{w}^{(1)}) * k^* + (w^{(1)})^2 * k + (w^{(1)})^2 * k^*] \qquad (3.9)$$

We denote the sum of the first two terms in the right-hand side of (3.9) by $h^{(1)}$, the sum of the two next following terms by j_1 and the sum of the last two terms by j_2. We apply Lemma 2.3 to j_1 and obtain $j_1 = -\lambda\,(w_{1/2}\,u) * (k + k^*) + \lambda i\,(w_{1/2}\,v) * (k - k^*)$

By virtue of (3.1), (3.2), and the first of relationships (3.8)

$$j_1 = i\lambda\,(u_{1/2}\,v) * L_{21} + O\,[(\rho^{-3/2+\varepsilon} + \rho^{-1/2-\gamma+\varepsilon})\,\sigma^{-1/2}(\zeta)] \qquad (\delta\,(u^{(1)}) = \gamma) \qquad (3.10)$$

Let us assume that $\gamma < 1$. Similarly

$$\lambda\,(u_{1/2}\,v) * L_{21} = O\,[\rho^{-1+\varepsilon} e^{\mu\,(\xi-\rho)} + \rho^{-3/2+\varepsilon}\,\sigma^{-1/2}(\zeta)] \qquad (3.11)$$

where $\vartheta\lambda$ with $\vartheta < 1$ can be taken for μ. We have

$$j_2 = -\lambda\,(vu^{(1)}) * (k - k^*) + {}^1/_2\,\lambda i\,[v^2 - (u^{(1)})^2] * (k + k^*)$$

Applying Lemma 2.2, we obtain

$$j_2 = i\lambda\,(vu^{(1)}) * L_{21} + O\,[(\rho^{-(1+1/2\gamma)+\varepsilon} + \rho^{-(1/2+\gamma)+\varepsilon})\,\sigma^{1/2(1-2\gamma+\varepsilon)}(\zeta) + \rho^{-2\gamma+\varepsilon}] \qquad (3.12)$$

Similarly

$$\lambda\,(vu^{(1)}) * L_{21} = O\,[\rho^{-(1/2+\gamma)+\varepsilon} + \rho^{-1/2(1+\gamma)+\varepsilon}\,\sigma^{1/2\gamma+\varepsilon}(\zeta)] \qquad (3.13)$$

By Lemma 2.3

$$w_{1/2}^2 * (k + k^*) = O\,(\rho^{-1}\,\sigma^{-1/2}(\zeta)) \qquad (3.14)$$

Since $\delta(f^{(1)}) \geqslant 1$, from the adduced estimates follows the inequality $\gamma \geqslant \min[1, {}^1/_2(1+\gamma)]$, which contradicts the assumption of $\gamma < 1$. Hence, $\gamma \geqslant 1$, Q.E.D.

The results presented in the form of propositions (3.1)–(3.3) appear in the paper by Smith [6], but his proof differs from that given here.

From (3.10) and (3.12) we have

$$v = \operatorname{Re} h^{(1)} + O\,[\rho^{-3/2+\varepsilon}\sigma_2^{-1/2+\varepsilon}(\zeta)] \qquad (3.15)$$

Hence, if the asymptotics of convolution (3.14) is found, it becomes possible to determine the principal term of the asymptotics of function v. The computation of the asymptotics of such integrals is a somewhat complicated and precise process. Here we present only the final results. By virtue of definition of functions k and k^*

$$w_{1/2}^2 * (k + k^*) = w_{1/2}^2(z) * \left(\frac{\partial l_0(\zeta - z)}{\partial\bar{\zeta}} + \frac{\partial m_0(\zeta - z)}{\partial\zeta}\right) + \lambda^{-1} w_{1/2}^2(z) * (\zeta - z)^{-2}$$

The integrals are taken here in the meaning of the Cauchy principal value. We denote the first and second terms in the right-hand side (of this equation) by $I_{1/2}$ and $J_{1/2}$, respectively. We have

$$I_{1/2}(z) = \frac{2ia_{1/2}^2}{\lambda r} e^{\lambda(x-z)} \left\{\left[2i\lambda\,(r - x) + \frac{y}{r}\left(\lambda\,(r - x) - \frac{1}{2}\right)\right] \int_0^1 e^{-\lambda(r-x)v^2} dv - \frac{1}{2}\left(i + \frac{y}{r}\right) e^{\lambda(x-r)}\right\} + r^{-3/2}\,[C_1 y + C_2 + C_3\,(r - x)]\,e^{\lambda(x-r)} + O\,(r^{-2} e^{\mu(x-r)})$$

where C_1, C_2 and C_3 are certain constants whose exact value is unessential in this context. Prior to adducing the formula for $J_{1/2}$ we shall make the following stipulations.

We cut plane z along the half-axis $x > 0$ (inside the trail!).

Function $(-z)^{1/2}$ is single-valued in the slit plane, and we take that branch of the root which is positive for $z < 0$. We have

$$J_{1/2}(z) = -\lambda^{-1}a_{1/2}^2\left[\frac{\bar z}{r^2}e^{2\lambda(x-r)} + \frac{(-z)^{-3/2}}{2\sqrt{\lambda}}\int_0^\tau e^{-t}\frac{dt}{\sqrt{t}}\right] + \sum_{k=1}^{\infty} C_k' z^{-k} \qquad (\tau = 2\lambda(r-x))$$

and the series converges in the neighborhood of point $z = \infty$. It is readily seen that, as long as $z \neq 0$, function

$$(-z)^{-3/2}\int_0^\tau e^{-t}\frac{dt}{\sqrt{t}}$$

and all of its derivatives are not subject to discontinuities along the slit.

From these expressions follows that

$$w_{1/2}^2 * (k+k^*) = \frac{a_{1/2}^2}{\lambda}\left\{\frac{2i}{r}\left[2i\lambda(r-x) + \frac{y}{r}\left(\lambda(r-x) - \frac{1}{2}\right)\right]\times\right.$$

$$\left.\times\, e^{\lambda(x-r)}\int_0^1 e^{-\lambda(r-x)v^2}dv - \frac{(-z)^{3/2}}{2\sqrt{\lambda}}\int_0^\tau e^{-t}\frac{dt}{\sqrt{t}}\right\} + (C_1 y + C_2 + C_3(r-x))\times$$

$$\times\frac{e^{\lambda(x-r)}}{r^{3/2}} + \sum_{k=1}^{\infty}\frac{C'_k}{z^k} + O(r^{-2}e^{\mu(x-r)}) \qquad (3.16)$$

Hence

$$\operatorname{Re} h^{(1)}(z) = \operatorname{Re}\left(a_1\frac{y}{r^{3/2}}e^{\lambda(x-r)} + \frac{b_1}{z}\right) + O(r^{-3/2})$$

$$\operatorname{Im} h^{(1)}(z) = \operatorname{Im}\left(a_1\frac{y}{r^{3/2}}e^{\lambda(x-r)} + \frac{b_1}{z}\right) -$$

$$-\frac{2\lambda a_{1/2}^2}{r}(r-x)\,e^{\lambda(x-r)}\int_0^1 e^{-\lambda(r-x)v^2}dv + O(r^{-3/2}) \qquad (3.17)$$

Stipulating

$$V_1(z) = \operatorname{Re}\left(a_1\frac{y}{r^{3/2}}e^{\lambda(x-r)} + \frac{b_1}{z}\right) \qquad (3.18)$$

from (3.15) and (3.17) we obtain

$$v = V_1(z) + O\left[r^{-3/2} + r^{-3/2+\varepsilon}\sigma^{-1/2+\varepsilon}(z)\right] \qquad (3.19)$$

4. Let $L(z) = L^{(1)}(z) + L^{(2)}(z)$ be one of the functions L_{lm} with $L^{(1)}(z)$ being that of the $L(z)$ components which exponentially decreases outside the trail. Let $\varphi(z)$ satisfy inequality $\quad |\varphi(z)| < (|\log r| + 1)^{\beta_0} r^{-\beta}\sigma^{-\gamma}(z)$

We set $\gamma_1 = \min(\gamma, 1/2)$ and $\gamma_2 = \min(\gamma, 1)$ and denote by $\varkappa$ an arbitrary quantity in the interval $(0, \infty)$. We introduce functions ζ

$$\Delta^{(1)} = \Delta_{\beta-\gamma,1} + \Delta_{2\gamma,2}\Delta_{\beta-\gamma_2,2}, \qquad \Delta^{(3)} = \Delta_{2\gamma,2}\Delta_{\beta+\gamma_2,3}$$

$$\Delta^{(2)} = \Delta_{2\gamma,1}\Delta_{\beta-\gamma_1,2} + \Delta_{2\gamma,2}\Delta_{\beta-\gamma_2,2}, \qquad \Delta^{(4)} = \Delta_{2\gamma,1}\Delta_{\beta+\gamma_1,3}$$

We set

$$\Omega(\zeta) = \rho^{-\alpha}\varphi_0(\zeta)\left[\Delta^{(1)}\rho^{1+1/2(\beta-\gamma_2)}\sigma^{1/2(1-\beta-\gamma)}(\zeta) + \Delta^{(2)}\rho^{3/2}\sigma^{-\gamma}(\zeta) + \Delta_{2\gamma,1}\rho^{2-\gamma_1}\sigma^{-\varkappa}(\zeta)\right] +$$

$$+\,\Delta^{(3)}\log^{\beta_0}\rho\,\rho^{-\alpha-1/2}\sigma^{-\varkappa}(\zeta) + (\Delta^{(4)} + \log\rho)\,\varphi_0(\zeta)\,\sigma^{-\gamma_1}(\zeta) \qquad (4.1)$$

Lemma 4.1. If $2 < \beta + \gamma_1 \leqslant 3$, then

$$J(\zeta) = (\varphi * L)(\zeta) = L(\zeta)\,\frac{1}{\pi}\int_G \varphi(z)\,dxdy + O(\Omega(\zeta))$$

Proof of this Lemma is omitted for the same reasons as given for Lemmas 2.1–2.3.

5. Let us determine the first order terms of the asymptotics of function $u(\zeta)$ by extending the reasoning of Sect. 3. Using formula (3.19) we transform the first term in formula (3.11) and then apply Lemma 4.1. We have

$$j_1 = i\lambda (u_{1/2} V_1) * L_{21} + i\lambda A_0 L_{21} + O[\rho^{-3/2+\varepsilon} \sigma^{-1/2}(\zeta)] \tag{5.1}$$

We carry out a similar operation on j_2. Then

$$j_2 = i\lambda (tu^{(1)}) * L_{21} + i\lambda A_1 L_{21} + O[\rho^{-5/4+\varepsilon} \sigma^{-3/4+\varepsilon}(\zeta) + \rho^{-3/2+\varepsilon} \sigma^{-1/2+\varepsilon}(\zeta)] \tag{5.2}$$

$$t(\zeta) = \operatorname{Re} b_1 \zeta^{-1}$$

Substituting (5.1) and (5.2) into (3.9) and taking the imaginary part, we obtain

$$u^{(1)} = \Phi_0 + \Phi_1 + \Phi_2, \qquad \Phi_0 = \operatorname{Im} h^{(1)} + \lambda A_2 L_{21} + \lambda (u_{1/2} V_1) * L_{21}$$

$$\Phi_1 = \lambda (tu^{(1)}) * L_{21} \quad (A_2 = A_0 + A_1) \tag{5.3}$$

with Φ_2 – the sum of residual terms expressed by

$$\Phi_2 = O[\rho^{-5/4+\varepsilon}\sigma^{-3/4+\varepsilon} + \rho^{-3/2+\varepsilon}\sigma^{-1/2+\varepsilon}] \tag{5.4}$$

Assuming

$$\psi(\zeta) = u_{1/2}(\zeta)\left(\bar{a}_1 \frac{y}{r^{3/2}} e^{\lambda(x-r)} + \frac{\bar{b}_1}{\bar{\zeta}}\right)$$

we obtain

$$(u_{1/2} V_1) * L_{21} = \operatorname{Re}(\psi * L_{21})$$

To facilitate the calculation of the last convolution we note that by Lemma 2.3

$$\psi * L_{21} = i\psi * (k - k^*) + O[\rho^{-3/2} \log \rho\sigma^{-1/2}(\zeta)]$$

$$\psi * (k - k^*) = \psi(z) * \left(\frac{\partial l_0(\zeta - z)}{\partial \bar{\zeta}} - \frac{\partial m_0(\zeta - z)}{\partial \zeta}\right) - \frac{\psi(z)}{\lambda} * \frac{1}{(\zeta - z)^2}$$

where the integrals are taken in the meaning of the Cauchy principal value.

We denote the first and second terms in the right-hand part by I_1 and J_1, respectively. Omitting intervening calculations, we note that

$$I_1(z) = -ia_{1/2}\bar{b}_1 \frac{y \log r}{r^{3/2}} e^{\lambda(x-r)} - a_{1/2}\bar{a}_1 i e^{\lambda(x-r)} [2(1 -$$

$$- \frac{x}{r})\int_0^1 e^{-(r-x)s^2} ds - \frac{1}{\lambda r} e^{\lambda(x-r)}] + iC_1 \frac{y}{r^{3/2}} e^{\lambda(x-r)} + O(r^{-3/2} \log r)$$

where C_1 is a certain real constant. Calculations yield

$$J_1(z) = O(r^{-3/2})$$

From these relationships and from (3.17) follows that

$$\Phi_0(z) = \lambda a_{1/2} \operatorname{Re} b_1 \frac{y \log r}{r^{3/2}} e^{\lambda(x-r)} + \operatorname{Im}\left(a_1^* \frac{y}{r^{3/2}} e^{\lambda(x-r)} + \frac{b_1}{z}\right) +$$

$$+ 2a_{1/2}(\operatorname{Re} a_1 - a_{1/2}) \frac{\lambda(r - x)}{r} e^{\lambda(x-r)} \int_0^1 e^{-\lambda(r-x)s^2} ds -$$

$$- \frac{a_{1/2} \operatorname{Re} a_1}{r} e^{2\lambda(x-r)} + O(r^{-3/2} \log r) \tag{5.5}$$

Proposition 5.1. Function $\Phi_1(z)$ satisfies relationship

$$\Phi_1(z) = AL_{21}(z) + O\,[r^{-3/2+\varepsilon}\sigma^{-1/2+\varepsilon}(z) + r^{-3/4+\varepsilon}\sigma^{-3/4+\varepsilon}(z)]$$

Proof. By virtue of (5.3)

$$\Phi_1 = \lambda\,(tu^{(1)}) * L_{21} = \lambda\,[(t\Phi_0) * L_{21} + (t\Phi_1) * L_{21} + (t\Phi_2) * L_{21}]$$

We substitute in the convolution $(t\Phi_0) * L_{21}$ the right-hand side of (5.5) for Φ_0 and, as the result, obtain a sum of convolutions to all of which, except one, Lemma 4.1 is applicable.

Thus

$$\lambda\,(t\Phi_0) * L_{21} = [(\mathrm{Im}\ b_1 z^{-1})\,t(z)] * L_{21}(\zeta - z) + A_3 L_{21}(\zeta) + O\,[\rho^{-3/4}\sigma^{-3/4}(\zeta)\log\rho]$$

By Proposition 3.3

$$u_1(z) = O\,(r^{-1+\varepsilon})$$

Hence by Lemma 2.2

$$\Phi_1(z) = O\,[r^{-1+\varepsilon}\,\sigma^{-1/2+\varepsilon}(z)]$$

and consequently by Lemma 4.1

$$\lambda\,(t\Phi_1) * L_{21} = A_4 L_{21}(\zeta) + O\,[\rho^{-3/4+\varepsilon}\sigma^{-3/4+\varepsilon}(\zeta)]$$

Finally, by virtue of (5.4) and Lemma 4.1

$$\lambda\,(t\Phi_2) * L_{21} = A_5 L_{21}(\zeta) + O\,[\rho^{-3/2+\varepsilon}\,\sigma^{-1/2+\varepsilon}(\zeta)]$$

Since

$$[\mathrm{Im}\,(b_1 z^{-1})\,t(z)] * L_{21}(\zeta - z) = \mathrm{Im}\,[{}^1/_2 b_1^{\,2} z^{-2} * L_{21}(\zeta - z)]$$

and it is not difficult to estimate the last convolution and find it to be $O\,(r^{-2}\log r)$, the proposition is proved.

It follows from this that the principal term of function $u^{(1)}$ differs from that of Φ_0 by a value of the form of AL_{21}, and is consequently determined by expression (5.5) but with a certain constant other than a_1^*. We shall denote this new constant also by a_1^*, since this will not result in any confusion.

6. Let us consider the question of differentiation of the derived asymptotic formulas. This question reduces specifically to the evaluation of the residue of the asymptotic formulas for $\partial w / \partial z$ and $\partial w / \partial \bar{z}$ when the principal terms are obtained by differentiation of the principal terms of function w. We note that for small $|z|$

$$k(z) = -\frac{1}{2\bar{z}} + \dots, \qquad k^*(z) = -\frac{\bar{z}}{2z^2} + \dots$$

where dots denote terms containing only a logarithmic singularity. From this by virtue of known theorems follows that function

$$\varphi_1(\zeta) = \frac{1}{\pi}\int\limits_{|z|\leqslant 1} \varphi(z)\,k(\zeta - z)\,dxdy \tag{6.1}$$

satisfies the inequality

$$|\varphi_1(\zeta + h) - \varphi_1(\zeta)| < C\max|\varphi|\,|h|\log\frac{1}{|h|}$$

if

$$|h| \leqslant 1$$

A similar statement is, also, valid for integrals with kernel $k^*(z)$.

To derive the asymptotics of function $w(\zeta + h) - w(\zeta)$ it would be necessary to repeat the reasoning of Sects. 3 and 5. However, since that reasoning is independent of the specific form of kernels $L_{lm}(z)$, it will remain valid also for kernels $L_{lm}(z + h) - L_{lm}(z)$, except that now the kernels satisfy inequality

$$|L_{lm}(z+h)-L_{lm}(z)|<C|h|[r^{-\alpha lm-1/2}e^{\mu(x-r)}+(\lambda r)^{-8}]$$

and not inequality (3.1), when

$$|z|\geqslant 1$$

It will be readily seen that the ancillary Lemmas of Sect. 2 and Lemma 4.1 remain valid, if one takes into consideration the remark about function (6.1). Hence

$$w(\zeta+h)-w(\zeta)=w_1(\zeta+h)-w_1(\zeta)+O(|h|\log\frac{1}{|h|}\rho^{-7/4+\varepsilon}) \tag{6.2}$$

If the Lipschitz condition

$$|\varphi(z_1)-\varphi(z_2)|\leqslant M_\alpha|z_1-z_2|^\alpha$$

is satisfied by function $\varphi(z)$ in (6.1), then $\varphi_1(\zeta)$ is differentiable, and

$$\left|\frac{\partial\varphi_1}{\partial\bar\zeta}\right|+\left|\frac{\partial\varphi_1}{\partial\zeta}\right|\leqslant C(\max|\varphi|+M_\alpha)$$

A similar statement is also valid for integrals with kernel k^*. With the use of the estimate (6.2) we can differentiate formula (3.9) and repeat the subsequent reasoning. As the result we obtain

$$\frac{\partial w(\zeta)}{\partial\bar\zeta}=\frac{\partial w_1(\zeta)}{\partial\bar\zeta}+O(\rho^{-7/4+\varepsilon}),\qquad \frac{\partial w}{\partial\zeta}=\frac{\partial w_1}{\partial\zeta}+O(\rho^{-7/4+\varepsilon}) \tag{6.3}$$

From this ensue the following propositions.

Proposition 6.1. If conditions (1.1) and (1.3) are satisfied, vortex ω satisfies inequality

$$|\omega(\zeta)|\leqslant C\rho^{-1} \tag{6.4}$$

Proof. Since $\omega(\zeta)=2\,\partial w/\partial\bar\zeta$, (6.4) follows by virtue of (6.3).

Let us consider the question of determining constants $a_{1/2}$, a_1 and b_1 in the asymptotic formula. These are not independent owing to certain interrelationships imposed on them by the continuity equation. By virtue of (6.3), (3.19) and (5.5)

$$\frac{\partial u}{\partial x}=-\frac{a_{1/2}}{2}\left(1-\frac{\lambda y^2}{r}\right)\frac{e^{\lambda(x-r)}}{r^{3/2}}+O(r^{-7/4+\varepsilon})$$

$$\frac{\partial v}{\partial y}=\operatorname{Re}a_1\left(1-\frac{\lambda y^2}{r}\right)\frac{e^{\lambda(x-r)}}{r^{3/2}}+O(r^{-7/4+\varepsilon})$$

and, therefore, the continuity equation implies

$$\operatorname{Re}a_1=1/2\ a_{1/2} \tag{6.5}$$

Proposition 6.2. The relationship

$$\operatorname{Im}b_1=-\frac{a_{1/2}}{\sqrt{2\pi\lambda}} \tag{6.6}$$

is valid.

Proof. The continuity equation and conditions along the body imply

$$\operatorname{Re}\int_C w(z)\,dz=0$$

whatever the closed contour C in region G.

Let

$$C=\{z:|z|=R\},\qquad R\uparrow\infty$$

We then obtain

$$\operatorname{Re}\left[ia_{1/2}\int_{-\pi}^{\pi}e^{\lambda R(\cos\theta-1)}R^{1/2}e^{i\theta}\,d\theta+2\pi ib_1\right]+O(R^{-1/4+\varepsilon})=0$$

Hence for $R\to\infty$ we have (6.6).

By virtue of (3.19), (5.5) and (6.5) we have the asymptotic formula

$$w(z) = ia_{1/2}\left(1 + \lambda \operatorname{Re} b_1 \frac{y \log r}{r}\right) r^{-1/2} e^{\lambda(x-r)} +$$
$$+ a_1 \frac{y}{r^{3/2}} e^{\lambda(x-r)} + \frac{b_1}{z} - ia_{1/2}^2 \frac{e^{\lambda(x-r)}}{r} \left[\lambda(r-x)\int_0^1 e^{-\lambda(r-x)s^2} ds + \right.$$
$$\left. + {}^1/_2 e^{\lambda(x-r)}\right] + \Omega(z) \tag{6.7}$$

where

$$\operatorname{Re}\Omega(z) = 0\,[r^{-3/2} + r^{-3/2+\varepsilon}\sigma^{-1/2+\varepsilon}(z)]$$
$$\operatorname{Im}\Omega(z) = O\,[r^{-3/2}\log r + r^{-5/4+\varepsilon}\sigma^{-3/4+\varepsilon}(z) + r^{-3/2+\varepsilon}\sigma^{-1/2+\varepsilon}(z)] \tag{6.8}$$

Summarizing the obtained results, we come to the theorem as follows.

Theorem 6.1. If conditions (1.1) and (1.2) are satisfied, there exists for the complex velocity the asymptotic formula (6.7) with the residual term (6.8).

7. Let us pass to the evaluation of the attenuation of the vortex outside the trail. For this we shall consider relationship (1.3) as the integral equation of function $\omega(z)$.

First, we shall establish an ancillary proposition. Let function $\psi(z)$ be continuous for $z \neq 0, \infty$ and for $0 < r < \infty$ satisfy the inequalities

$$0 \leqslant \psi(z) \leqslant C_0 r^{-\gamma}, \qquad 0 < \gamma < 1 \tag{7.1}$$

$$\psi(\zeta) \leqslant \rho^{-\gamma} e^{\mu(\xi-\rho)} + A\,[r^{-1}\psi(z)] * L(\zeta - z) \tag{7.2}$$

$$(L(z) = r^{-1} e^{\mu(x-r)}, \quad A = \text{const})$$

Proposition 7.1. Inequality (7.2) with condition (7.1) implies

$$\psi(\zeta) \leqslant B_0 \rho^{-\gamma} e^{\mu_0(\xi-\rho)} \tag{7.3}$$

where B_0 and μ_0 are suitable constants.

Proof. We set $\mu = 2\mu_1 + \mu_2$, $\mu_j > 0$, $j = 1, 2$, and assume that for $s \leqslant n$, where n is an integer or a half-integer,

$$\psi(z) \leqslant C_0 B^s \Gamma(s+1)\,\sigma^{-s}(\mu_1 z)\, r^{-\gamma} \tag{7.4}$$

This inequality is satisfied at $s = 0$. We shall prove that with a suitable selection of constant B it will also be satisfied for $s \leqslant n + 1/2$ and, consequently, for all integral and half-integral n. Setting $\tau = \rho - \zeta$, we introduce sets

$$G_0 = \left\{z : r - x < \frac{1}{2}\tau\right\}, \quad G_n = \left\{z : \tau\left(\frac{1}{2} + \frac{[n]-1}{2n}\right) \leqslant r - x\right\} \tag{7.5}$$
$$G_k = \left\{z : \tau\left(\frac{1}{2} + \frac{k-1}{2n}\right) \leqslant r - x < \tau\left(\frac{1}{2} + \frac{k}{2n}\right)\right\}, \quad k = 1, 2, \ldots, [n] - 1$$

Noting that

$$\xi - x - |\zeta - z| \leqslant \xi - \rho + r - x$$

we obtain

$$e^{\mu(\xi-x-|\zeta-z|)} \leqslant e^{\mu_2(\xi-x-|\zeta-z|)} e^{-\tau[1-(k/n)]\mu_1}$$
$$z \in G_k, \; 0 \leqslant k \leqslant [n] - 1$$

It is easy to verify that for $t > 0$ and $m > 0$,

$$e^t \geqslant \frac{t^m}{\Gamma(m+1)}$$

Hence, for $z \in G_k$, $0 \leqslant k \leqslant [n] - 1$

$$e^{\mu(\xi - x - |\zeta - z|)} \leqslant e^{\mu_2(\xi - x - |\zeta - z|)} \frac{\Gamma(n-k+1)}{[\mu_1 \tau (1 - (k/n)) + 1]^{n-k}}$$

For $z \in G_k$, $0 \leqslant k \leqslant [n] - 1$ by inductive proposition

$$\psi(z) \leqslant C_0 B^k \Gamma(k+1) [{}^1/_2 \mu_1 \tau (1 + (k-1)/n) + 1]^{-k} r^{-\gamma}$$

If $z \in G_n$, then

$$\psi(z) \leqslant C_0 B^n \Gamma(n+1) \left[\tau\left(1 - \frac{n+1-[n]}{2n}\right) + 1\right]^{-n} r^{-\gamma}$$

It is readily seen that

$$\left[\mu_1 \tau \left(1 - \frac{n - [n] + 1}{2n}\right) + 1\right]^{-n} \leqslant \left(1 - \frac{1}{n}\right)^{-n} (\mu_1 \tau + 1)^{-n}$$

Hence

$$\psi(z) \leqslant 4 C_0 B^n \Gamma(n+1) (\mu_1 \tau + 1)^{-n} r^{-\gamma} \qquad (n \geqslant 2,\ z \in G_n) \tag{7.6}$$

If $n < 2$, we take the complement to G_0 for G_n, and then estimate (7.6) will be valid. We set

$$M_n = \max\{M_1^*, M_2^*\}$$

$$M_1^* = \max \frac{\Gamma(n-k+1)\Gamma(k+1) B^k}{[\mu_1 \tau (1 - k/n) + 1]^{n-k} [{}^1/_2 \mu_1 \tau (1 + k/n) + 1]^k}$$

$$M_2^* = 4B^n \frac{\Gamma(n+1)}{(\mu_1 \tau + 1)^n} \qquad (0 < k \leqslant [n] - 1)$$

Taking the integration interval of convolution (7.2) as the sum of intervals G_k and then applying in the interval over G_k the corresponding inequality, we obtain

$$[r^{-1} \psi(z)] * L(\zeta - z) < M_n (r^{-1-\gamma}) * L_1(\zeta - z), \qquad L_1(z) = r^{-1} e^{\mu_2 (x - r) + 1}$$

The last convolution by Lemma 2.2 does not exceed

$$C \rho^{-1/_2 (1+\gamma)} \sigma^{-1/_2 \gamma} (\mu_1 \zeta)$$

Hence by virtue of (7.2)

$$\psi(\zeta) < \rho^{-1} e^{\mu(\xi - \rho)} + A C_1 M_n \rho^{-\gamma} \sigma^{-1/_2} (\mu_1 \zeta), \qquad C_1 = C C_0$$

Constant C depends only on μ_1, μ_2 and γ. Let us find the upper limit of M_n. Clearly, we can assume $n \gg 1$ and $k \gg 1$. It is readily seen that

$$\left[\mu_1 \tau \left(1 - \frac{k}{n}\right) + 1\right]^{n-k} \left[\frac{\mu_1 \tau}{2}\left(1 + \frac{k-1}{n}\right) + 1\right]^k \geqslant (\mu_1 \tau + 1)^n \left(1 - \frac{k}{n}\right)^{n-k} \left(\frac{1}{2} + \frac{k-1}{2n}\right)^k$$

Taking this inequality into consideration and using inequalities

$$m^{m + 1/_2} e^{-m} \sqrt{2\pi} < \Gamma(m+1) < m^{m+1/_2} e^{-m} \sqrt{2\pi e}$$

we obtain

$$\frac{\Gamma(n-k+1)\Gamma(k+1)}{[\mu_1 \tau (1 - k/n) + 1]^{n-k} [{}^1/_2 \mu_1 \tau (1 + (k-1)/n) + 1]^k} <$$

$$< \frac{\Gamma(n + {}^3/_2)}{(\mu_1 \tau + 1)^n} \sqrt{2\pi l^3} \left(\frac{k}{n}\right)^{k + 1/_2} \left(1 - \frac{k}{n}\right)^{1/_2} \left(\frac{1}{2} + \frac{k-1}{2n}\right)^{-k} \left(1 + \frac{1}{2n}\right)^{-(n+1)}$$

But

$$\left(\frac{k}{n}\right)^{k + 1/_2} \left(1 - \frac{k}{n}\right)^{1/_2} \left(\frac{1}{2} + \frac{k-1}{2n}\right)^{-k} \left(1 + \frac{1}{2n}\right)^{-(n+1)} <$$

$$< \left(\frac{k/n}{{}^1/_2 (1 + k/n)}\right)^k \left(\frac{k}{n}\right)^{1/_2} \left(1 - \frac{k}{n}\right)^{1/_2} \left(1 + \frac{1}{n+k-1}\right)^k \left(1 + \frac{1}{2n}\right)^{-(n+1)} < \frac{1}{2}$$

Hence

$$M_n < 2B^n \Gamma(n + {}^3/_2)(\mu_1\tau + 1)^{-n}$$

Consequently

$$\psi(\zeta) < \rho^{-1} e^{\mu(\xi-\rho)} + 2AC_1B^n\rho^{-\gamma}\Gamma\left(n + \frac{3}{2}\right)\sigma^{-n-1/2}(\mu_1\xi)$$

If we select constant B so that

$$e + 2A_1C_1B^n \leqslant C_0B^{n+1/2}$$

$$B \geqslant (2AC + (e / C_0B^n))^2$$

and apply to the first term the inequality (7.5) we obtain for $s = n + {}^1/_2$ the inequality (7.4).

Thus it is sufficient to set $B = 4AC$, since it can be always assumed that

$$e / C_0B^n \leqslant 1, \quad 2AC > 1$$

The validity of inequality (7.4) is thereby established for the general case. From (7.4) follows

$$\psi(\zeta) \leqslant C_0\rho^{-\gamma} \min_s \left[\frac{B^s\Gamma(s+1)}{\sigma^s(\mu_1\zeta)}\right] \leqslant C_2\rho^{-\gamma}\sigma^{1/2}(\mu_1\zeta)\exp\frac{-\mu_1(\rho-\xi)}{B}$$

From this follows (7.3), if $\mu_0 = \mu_1/2B$ is assumed.

N o t e. We can assume $\mu_1 = \mu_2 = \mu / 3$. Then

$$\mu_0 = \mu/24AC \tag{7.7}$$

and $C = C(\mu)$ if γ is fixed.

P r o p o s i t i o n 7.2. If the conditions of Proposition 7.1 are satisfied, then

$$\psi(\zeta) < C_\varepsilon\rho^{-\gamma}e^{(\mu-\varepsilon)(\xi-\rho)}$$

where $\varepsilon > 0$ is arbitrarily small, and C_ε depends on A, μ and ε.

P r o o f. The set M of those values of m for which inequality

$$\sup_{|\zeta|<\infty} [\rho^\gamma \psi(\zeta) e^{m(\rho-\xi)}] < \infty \tag{7.8}$$

is satisfied is, by Proposition 7.1, not empty, since $\mu_0 \in M$. We note that when $m_0 \in M$, then also $(0, m_0] \subset M$. Let $\mu^\circ = \sup m \quad (m \in M)$

We shall prove that $\mu^\circ \geqslant \mu$ by contradiction. We assume $\mu^\circ < \mu$. Let $\mu^* = \mu - \mu^\circ$ and $\varepsilon > 0$ be sufficiently small. Clearly $\mu^\circ - \varepsilon = m \in M$. We set

$$\psi_1(\zeta) = \psi(\zeta)\, e^{m(r-\xi)}$$

It then follows from (7.2) that

$$\psi_1(\zeta) \leqslant \rho^{-1} e^{(\mu^*+\varepsilon)(\xi-\rho)} + [r^{-1}\psi_1(z)] * L_1(\zeta - z)$$

$$L_1(z) = r^{-1}\exp[(\mu^* + \varepsilon)(x - r)]$$

Condition (7.1) assumes the form

$$0 \leqslant \psi_1(\zeta) \leqslant C_m\rho^{-\gamma}$$

Hence by Proposition 7.1

$$\psi_1(\zeta) \leqslant B_m\rho^{-\gamma}\exp[\mu_m(\xi - \rho)]$$

and according to (7.7) we have $\mu_m = (\mu^* + \varepsilon)[24AC(\mu^* + \varepsilon)]^{-1}$. Since for any $\varepsilon > 0$ constant $C(\mu^* + \varepsilon) < C^*$, hence for small ε the value $\mu_m + m > \mu^\circ$, while on the other hand $\mu_m + m \in M$, which is absurd. Thus $\mu^\circ \geqslant \mu$, Q. E. D.

To apply Proposition 7.2 to the asymptotics of the vortex we transform (1.3). By Theorem 6.1

$$|v(z)| < C_0r^{-1}, \; |u(z)| < C_1r^{-1/2}e^{\lambda(x-r)} + C_0r^{-1}$$

Applying again Proposition 6.1 and the readily established inequalities

$$\left|\frac{\partial l_0(z)}{\partial x}\right| < C_2 \left|1 - \frac{x}{r}\right| e^{\lambda(x-r)} \frac{\sqrt{r}+1}{r}, \quad \left|\frac{\partial l_0(z)}{\partial y}\right| < C_2 \frac{|y|}{r} e^{\lambda(x-r)} \frac{\sqrt{r}+1}{r} \tag{7.9}$$

we obtain

$$|\omega(\zeta)| < |j_0| + A_0\left(\frac{|\omega(z)|}{r}\right) * L_0(\zeta - z) + C_3 [r^{-3/2} e^{\lambda(x-r)}] * L_1(\zeta - z)$$

$$L_0(z) = \frac{\sqrt{r}+1}{r}\left[\frac{|y|}{r} + \left|1 - \frac{x}{r}\right|\right] e^{\lambda(x-r)}, \quad L_1(z) = \frac{\sqrt{r}+1}{r}\left|1 - \frac{x}{r}\right| e^{\lambda(x-r)}$$

Passing to elliptic coordinates, as was done in the proof of Lemma 2.3, it can be easily shown that

$$[r^{-3/2} e^{\lambda(x-r)}] * L_1(\zeta - z) < C\left[\frac{\log \lambda\rho}{(\lambda\rho)^{1/2}}\left(\frac{1}{\lambda\rho} + 1 - \frac{\xi}{\rho}\right) + \frac{|\eta|}{\lambda\rho}\right] e^{\lambda(\xi-\rho)} \tag{7.10}$$

Hence from (1.4), (7.9) and (7.10) follows

$$|\omega(\zeta)| < C_\mu \frac{e^{\mu(\xi-\rho)}}{\rho} + A_\mu\left[\frac{|\omega(z)|}{r}\right] * L(\zeta - z)$$

$$\mu = \lambda - \varepsilon, \qquad L(z) = \frac{1}{r} e^{\mu(x-r)}, \qquad \varepsilon > 0$$

Here ε is an arbitrary quantity. Assuming

$$\psi(z) = C_\mu^{-1} |\omega(z)|, \text{ when } z \in G \qquad \psi = 0, \text{ when } z \in S$$

we obtain for $\psi(z)$ the inequality (7.2), and by virtue of (6.4)

$$\psi(z) < Cr^{-\gamma}, \qquad \gamma < 1$$

Thus from Proposition 7.2 we obtain the following fundamental inequality

$$|\omega(z)| < \frac{C}{r^{\varkappa}} e^{(\lambda-\varepsilon)(x-r)} \tag{7.11}$$

where $\varepsilon > 0$ is arbitrary and small, and $\varkappa < 1$. Constant C depends on λ, ε and $\varkappa$.

8. Let us define more precisely the asymptotic formula for the vortex. Two terms of the asymptotics of function $\omega(z)$ can be obtained from (6.3), but with residue $O(\rho^{-7/4+\varepsilon})$ which we know must exponentially decrease outside the trail. To avoid lengthy calculations we shall consider the simplest case in which only the first term of the asymptotics is retained.

Lemma 8.1. Let

$$L(z) \equiv L^{(1)}(z), \quad |\varphi(z)| < (|\log r| + 1)^{\beta_0} r^{-\beta} e^{\mu(x-z)}$$

be satisfied in addition to the conditions of Lemma 4.1.

Then

$$J(\zeta) = A_0 L(\zeta) + O(\Omega(\zeta) e^{\mu_0(x-z)}) \qquad (\mu_0 < \mu) \tag{8.1}$$

Here $\Omega(\zeta)$ is given by expression (4.1) in which $\gamma_1 = 1/2$ and $\gamma_2 = 1$.
Proof of Lemma 8.1 is an exact repetition of that of Lemma 4.1.
From (1.6), (7.10) we have

$$\omega(\zeta) = j_0 + \frac{\lambda}{\pi}\int_G \omega v \frac{\partial l_0(\zeta - z)}{\partial y}\, dx dy + O\left(\frac{\log\rho}{\rho^{3/2}} e^{\mu(\xi-\rho)}\right)$$

$$(\mu < \lambda)$$

Applying Lemma 8.1 by virtue of (7.11) and taking into account that

$$j_0 = A_1 \frac{\eta}{\rho^{3/2}} e^{\lambda(\xi-\rho)} + O(\rho^{-3/2} e^{\mu(\xi-\rho)})$$

we obtain

$$\omega(\zeta) = A \frac{\eta}{\rho^{3/2}} e^{\lambda(\xi-\rho)} + e^{\mu(\xi-\rho)} O\left(\frac{\log \rho}{\rho^{3/2}} + \rho^{-1/2-\varkappa}\right)$$

This shows that $\varkappa = 1$ can be assumed in (7.11). Thus, repeating the reasoning for $\varkappa = 1$ and taking into account that by virtue of (6.3) $A = \lambda a_{1/2}$, we find

$$\omega(\zeta) = \lambda a_{1/2} \frac{\eta}{\rho^{3/2}} e^{\lambda(\xi-\rho)} + e^{\mu(\xi-\rho)} O\left(\frac{\log \rho}{\rho^{3/2}}\right) \tag{8.2}$$

Theorem 8.1. In conditions defined by Theorem 7.1 there exists relationship (8.2) in which μ is any quantity smaller than λ.

Repeating literally the reasoning of Sect. 6, we obtain for $\partial\omega/\partial\bar{\zeta}$ and $\partial\omega/\partial\zeta$ the following asymptotic formulas:

$$\frac{\partial\omega}{\partial\bar{\zeta}} = \frac{\partial\omega_1}{\partial\bar{\zeta}} + e^{\mu(\xi-\rho)} O\left(\frac{\log \rho}{\rho^2}\right), \quad \frac{\partial\omega}{\partial\zeta} = \frac{\partial\omega_1}{\partial\zeta} + e^{\mu(\xi-\rho)} O\left(\frac{\log \rho}{\rho^2}\right) \tag{8.3}$$

where ω_1 is the principal term in (8.2).

9. Let X and Y be the projections of the force acting on the body on axes x and y. Simple calculations yield

$$X + iY = -2\pi i \rho b_1 \tag{9.1}$$

We denote by Γ the limit of velocity circulation along the contour C when it tends to ∞. It is easily shown that

$$\operatorname{Re} b_1 = \frac{1}{2\pi}\Gamma \tag{9.2}$$

Theorem 9.1. In conditions of Theorem 6.1 lift is defined by formula

$$Y = -\rho u_\infty \Gamma \tag{9.3}$$

This theorem is an extention of the Zhukovskii (Joukowsky) theorem to the case of a viscous fluid. This theorem was obtained by Filon already in 1926, although without a rigorous proof.

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